On Metric Perturbations in Brane-World Scenarios

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Abstract

In this note we reconsider linearised metric perturbations in the one-brane Randall-Sundrum Model. We present a simple formalism to describe metric perturbations caused by matter perturbations on the brane and remedy some misconceptions concerning the constraints imposed on the metric and matter perturbations by the presence of the brane.

An interesting alternative to standard Kaluza-Klein compactification is to view our 4-dimensional Universe as a 3-brane embedded in a bigger space with large, or infinite extra dimensions, but such that matter fields are localised on the brane [1]-[12]. Considerable effort has been devoted to investigate possible observable consequences of such scenarios. For this one has to determine how the gravitational dynamics, including matter is affected by the extra dimensions in which gravity can propagate. The linearised metric perturbation in Brane-World Scenarios have been considered by numerous authors [8, 9], [13]-[24].

In [13, 16] the authors considered linear perturbations of a Minkowski brane in coordinates in which the four-dimensional metric is transverse-traceless. In this gauge the brane appears "bent" in the presence of matter, that is, the coordinates are discontinuous on the brane. In [17] and [22] the same analysis was carried out in alternative gauges for which the coordinates are continuous across the brane. On the other hand linear perturbations were analysed for an expanding Universe on the brane [14]-[24] and some cosmological consequences were analysed using a generalisation of the $3+1$ decomposition of the metric perturbations [25, 26, 27]. In this note we present a simple generalisation of the formalism [22] which allows us to relate and correct some of the results obtained within the different approaches for metric perturbations caused by

matter on a Minkowski brane. Concretely we find the general solution for linear metric perturbations generated by arbitrary matter perturbations on the brane in the axial gauge $H_{\mu 4}=0$. There is a degeneracy in the system of linear perturbation equations due to the remaining scalar scalar gauge-freedom. Exploring the space of solution for the linear perturbations we may identify various previous solutions found in different gauges. On the other hand we find that the so-called generalised longitudinal gauge for the scalar perturbations [20] can not be imposed for generic matter perturbations on the brane. We find the approach presented here very well suited for perturbations induced by matter on the brane, however, we do not address primordial perturbations in this paper. Also we leave the generalisation to expanding brane cosmologies for future work.

To describe the metric perturbations consider the Ansatz

$$\begin{aligned} ds^2 = g_{AB}dx^A dx^B &= e^{2A(z)}\left(\eta_{AB} + H_{AB}\right)dx^A dx^B \ , \\ A &= -\log(1+\kappa|z|) \ , \end{aligned} \tag{1}$$

which, in the absence of matter satisfies the background Einstein equations[1]

$$\mathcal{G}_{AB} = \Lambda g_{AB} + V\eta_{\mu\nu}\delta(z) \ , \tag{2}$$

provided

$$\Lambda = 6\kappa^2 \quad \text{and} \quad V = -6\kappa \ . \tag{3}$$

Next we consider the linearised Einstein equations for the perturbation H_{AB} (in the gauge $H_{\mu 4} = 0$) in the presence of matter localised on the brane. After a series of standard manipulations and Fourier transformation along the brane we end up with the system of equations

$$\begin{aligned} \frac{1}{2}p^2 H_{\mu\nu} - \frac{1}{2}H''_{\mu\nu} + \frac{1}{2}p_\mu p_\nu H_{44} - p_\mu p_{(\nu}H_{\mu)\lambda} + & \\ \frac{4\kappa^2}{(1+\kappa z)^2}\eta_{\mu\nu}H_{44} + \frac{\kappa}{2(1+\kappa z)}\left(H'\eta_{\mu\nu} - H'_{44}\eta_{\mu\nu} + 3H'_{\mu\nu}\right) &= 0 \\ -\frac{1}{2}p_\mu H' + \frac{1}{2}p^\lambda H'_{\lambda\mu} - p_\mu \frac{3\kappa}{2(1+\kappa z)}H_{44} &= 0 \\ \frac{1}{2}p^2 H_{44} - \frac{1}{2}H'' - \frac{\kappa}{2(1+\kappa z)}\left(H'_{44} - H'\right) + \frac{4\kappa^2}{(1+\kappa z)^2}H_{44} &= 0 \ , \end{aligned} \tag{4}$$

for $z > 0$, together with the jump conditions at $z=0$

$$-\frac{1}{2}[H'_{\mu\nu}] = \kappa H_{44}\eta_{\mu\nu} + T_{\mu\nu} - \frac{1}{3}T\eta_{\mu\nu} \ . \tag{5}$$

Here prime denotes the derivative alon z, $T_{\mu\nu}$ is the matter stress tensor and $p^2=p^\mu p_\mu$. Following [22] we substitute the general Ansatz for $H_{\mu\nu}$ and H_{44} perturbations on the brane

$$\begin{aligned} H_{\mu\nu} &= a(p,z)T_{\mu\nu} + b(p,z)T\eta_{\mu\nu} + c(p,z)p_\mu p_\nu T \\ H_{44} &= d(p,z)T \ , \end{aligned} \tag{6}$$

[1] In units with $8\pi G_5=1$; $A=0,\cdots,4$; $\mu=0,\cdots,3$.

Note that (6) is not the general solution to (4) as we can always add a solution of the homogeneous equations H^0_{AB} to (6). In what follows we will set $H^0_{AB}=0$, that is, we restrict ourself to perturbations generated by matter on the brane. Substitution of (6) into (4) then leads to the set of equations [22]

$$a' + 3b' + \frac{3\kappa}{1+\kappa z} d = 0 \tag{7}$$

$$p^2\left(a + 3b\right) - \frac{\kappa}{1+\kappa z}\left(a' - 3p^2 c'\right) = 0 \tag{8}$$

$$a'' - \frac{3\kappa}{1+\kappa z} a' - p^2 a = 0 \tag{9}$$

$$a + 2b - c'' + \frac{3\kappa}{1+\kappa z} c' + d = 0\ , \tag{10}$$

supplemented with the jump conditions

$$\frac{1}{2}[a'] = -1 \quad ; \quad \frac{1}{2}[b'] = \frac{1}{3} - \kappa d\ , \tag{11}$$

and c' is continuous across $z=0$. The $\mathbf{Z}_2$-symmetry $z \to -z$ implies that $c'|_0 = 0$.

It turns out that the system of equations (7)-(10) is degenerate. Here we relate this degeneracy with the remaining scalar gauge freedom by comparing the Ansatz (6) with the general Ansatz [20, 23].

$$g_{AB} = e^{2A(z)} \begin{pmatrix} -(1+2\Phi) & -W_{0i} & -W \\ -W_{0i} & (1-2\Psi)\delta_{ij} + 2p_i p_j E + 2p_{(i}F_{j)} + h_{ij} & W_{4i} \\ -W & W_{4i} & (1+2\Gamma) \end{pmatrix}\ , \tag{12}$$

where

$$W_{ai} = p_i B_a + S_{ai}\ , a = 0, 4\ , \tag{13}$$

and S_{ai} is divergence free $p^i S_{ai} = 0$. The diffeomorphisms ξ^A are also decomposed into divergence free 3-vectors ξ^i (2 degrees of freedom), and 3 scalars[2] ξ, ξ^0, ξ^5. In what follows we use the vector gauge degrees of freedom to set

$$S_{4i} = 0\ . \tag{14}$$

Under the remaining diffeomorphisms the different scalars in (12) transform as

$$\begin{aligned} &\delta\Psi = -A'\xi_4 \quad ; \quad \delta\Phi = p_0\xi_0 - A'\xi_4 \quad ; \quad \delta\Gamma = \xi_4' + A'\xi_4 \\ &\delta B_0 = \xi_0 - p_0\xi \quad ; \quad \delta B_4 = -\xi_4 + \xi' \quad ; \quad \delta E = \xi \quad ; \quad \delta W = p_0\xi_4 + \xi_0'\ . \end{aligned} \tag{15}$$

In order to compare the two Ansätze we further fix two scalar gauge degrees of freedom by imposing

$$B_4 = W = 0\ , \tag{16}$$

[2]Note, however constraint $\xi^5(y=0,x)=0$ in in the presence of a 3-brane at $z=0$

which can always be chosen for matter localised on the brane (see also [22]). To continue we compare the parametrisations (12) with (6) leading to the identifications

$$\begin{aligned} -2\Phi &= a\rho - bT + cp_0T \ , \\ S_{0i} &= aPe_i \ , \\ B_0 &= cp_0T \ , \\ -2\Psi &= ap + bT - \frac{1}{3}a\Delta\tilde{\pi} \ , \\ 2E &= a\tilde{\pi} + cT \ , \\ F_i &= a\pi_i \ , \\ h_{ij} &= a\bar{\pi}_{ij} \ , \\ 2\Gamma &= dT \ . \end{aligned} \tag{17}$$

Here, ρ and P are the density- and pressure perturbations on the brane and e_i is a transverse unit vector. The quantities $\tilde{\pi}, \pi_i$ and $\bar{\pi}_{ij}$ denote the scalar-, vector- and tensor part of the anisotropic stress and a, b, c, d are the solutions of the system of equations (7)-(10). To complete the comparison between the Ansatz (6) and the parametrisation (12) we now compare the jump conditions found with the two Ansätze. The Ansatz (6) together with the jump conditions (11) leads to

$$\begin{aligned} \Phi'|_{z=0} &= \frac{1}{3}\rho + \frac{1}{2}p - \kappa\Gamma \\ \Psi'|_{z=0} &= \frac{1}{6}\rho - \kappa\Gamma \\ S'_{0i}|_{z=0} &= -pe_i \\ F'_i|_{z=0} &= -\pi_i \\ h'_{ij}|_{z=0} &= -\bar{\pi}_{ij} \\ E'|_{z=0} &= -\frac{1}{2}\tilde{\pi} \\ B'_0|_{z=0} &= 0 \ . \end{aligned} \tag{18}$$

For $\tilde{\pi} = 0$ these jump conditions agree with those found in [20, 23] respectively, thus establishing the equivalence with the Ansatz (6) modulo homogeneous solutions of the linearised Einstein equations (7)-(10).

As explained above we still have one scalar gauge transformation available. This allows, for example, to bring the metric into the form $B_0 = 0$ while keeping $W = B_4 = 0$. This is achieved by

$$\xi = \frac{1}{2}cT \quad ; \quad \xi_4 = \xi' \quad ; \quad \xi_0 = -\frac{1}{2}cp_0T \ . \tag{19}$$

This is the solution presented in [22].

Next we discuss what happens if we try to fix the remaining scalar gauge freedom by imposing the so-called generalised longitudinal gauge $E = B_0 = B_4 = 0$ [20]. We do this by taking as a starting point the gauge $c = 0$. The longitudinal gauge is then obtained with

$$\xi = -\frac{1}{2}a\tilde{\pi} \quad ; \quad \xi_4 = \xi' \quad ; \quad \xi_0 = -\frac{1}{2}a\tilde{\pi} \ . \tag{20}$$

Note that in this gauge $W = -a' p_0 \tilde{\pi}$ no longer vanishes. More importantly,

$$\xi^4(z=0) = -\frac{1}{2} a' \tilde{\pi} = \frac{1}{2} \tilde{\pi} \neq 0 \ , \tag{21}$$

that is, this gauge is only compatible with having the brane at $z=0$ if $\tilde{\pi}=0$. This is the conclusion reached in [20]. However, as we can see from the above this conclusion is gauge dependent as it only holds in the longitudinal gauge chosen in [20][3].

Let us now see how the metric perturbation in a gauge with $H_{44}=0$ [9, 13, 16] are obtained in our approach. It is clear that this gauge corresponds to $d=0$. From (15) we see that $d=0$ can be achieved by

$$\xi_4' - \frac{\kappa}{(1+\kappa z)} \xi_4 = -\frac{d}{2} T(p) \ . \tag{22}$$

Using (7) we integrate this equation as

$$\xi_4 = \alpha_0(p)(1+\kappa z) + \frac{(1+\kappa z)}{6\kappa}(a+3b) T(p) \ , \tag{23}$$

where $\alpha_0(p)$ is an integration constant to be determined later. Without restricting the generality we can assume $c=0$ before performing the gauge transformation leading to $d=0$. In that case we can use (8) to integrate the constraint $\xi_4 = \xi'$ as

$$\xi = \alpha_1(p) + \frac{\alpha_0(p)}{2\kappa}(1+\kappa z)^2 + \frac{1}{6p^2} a(p,z) T(p) \ . \tag{24}$$

Here, the integration constant $\alpha_1(p)$ corresponds to the z-independent four dimensional gauge transformation which we can set to zero at present (see also [13]). The remaining integration constant $\alpha_0(p)$ is, in turn, determined by the requirement that $\xi_4(p,0)=0$ [22]. This then implies $\alpha_0(p) = \frac{1}{6p^2} T(p)$. Finally we obtain the function $c(p,z)$ using (17) and (15), i.e.

$$\delta c(p,z) p_0 T(p) = -\delta B_0 = 2 p_0 \xi \ , \tag{25}$$

up to a z-independent integration constant which we set to zero. Thus,

$$\delta c(p,z) = -\frac{1}{3p^2}\left(\frac{(1+\kappa z)^2}{2\kappa} + a(p,z)\right) \ . \tag{26}$$

Correspondingly, $\delta b(p,z)$ is given by

$$\delta b(p,z) = \frac{\kappa}{(1+\kappa z)} \frac{1}{3p^2} \left(1 + \kappa z + a'\right) \ . \tag{27}$$

The function $a(p,z)$ is gauge-invariant. Note, however, that in this gauge eqn. (7) implies that $a+3b$ is independent of z. On the other hand a' is bounded [22] and hence (8) implies that c' diverges quadratically as $z \to \infty$, as previously observed [13, 16, 22]. Here, the large z divergences of the linear perturbation H_{AB} in this gauge are directly read off from from (7)-(10). As explained in [13, 16, 22] in order to remedy this divergence we need to relax the condition that the brane be situated at

[3] We have been informed that this problem concerning the longitudinal gauge will also be discussed in [28].

$z=0$. This then allows us to impose the extra conditions $p^\nu H_{\mu\nu}=H^\mu_\mu=0$, that is, the Randall-Sundrum gauge. Indeed, as is clear from (6) these two conditions correspond to

$$b+\delta b+\delta c p^2 = a + 4(b+\delta b)+\delta c p^2 = 0 \ , \tag{28}$$

where we have used again that $c=0$ before the gauge transformation. Substituting the expressions (26) and (27) into (28) it is then easy to see that (28) is fulfilled for $\alpha_0=0$. The displacement of the brane, that is, $\xi_4|_{z=0}$ is then given by

$$\xi_4(p,0) = -\frac{1}{6p^2}T(p) \ , \tag{29}$$

in agreement with [13, 16]. In this gauge, $c'(p,0)T(p)$ measures the displacement of the brane. On the other hand b and c are expressed algebraically in terms of a and hence b and c are bounded functions of z. Note, however, that this gauge violates the consistency condition $c'(p,0)=0$. This inconsistency is resolved by noting that in this gauge the coordinates are discontinuous on the brane, so that the Einstein equations (4) are modified.

Note added: After submitting the original version of this note we became aware of the latest version of [22] which also discusses gauge aspects in the system dicussed here. We would like to thank Z. Kakushadze for pointing this out to us.

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References

[1] V.A. Rubakov and M. Shaposhnikov, Phys. Lett. **B125** (1983) 136; ibid 139; K. Akama, "Pregeometry", in Springer Lecture Notes in Physics, 176, (1982), hep-th/0001113.

[2] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B429** (1998) 263; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B436** (1998) 257.

[3] G. Shiu and S. H. H. Tye, Phys. Rev. **D58** (1998) 106007

[4] R. Sundrum, Phys. Rev. **D59** (1999) 085009

[5] Z. Kakushadze and S. H. H. Tye, Nucl. Phys. **B548** (1999) 180.

[6] S. Nussinov and R. Shrock, Phys. Rev. **D59** (1999) 105002.

[7] A. Donini and S. Rigolin, Nucl. Phys. **B550** (1999) 59.

[8] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370.

[9] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 4690.

[10] M. Gogberashvili, Mod. Phys. Lett. **A14** (1999) 2025, hep-th/9904383.

[11] N. Nicolai, Phys. Lett. **B150** (1985) 347.

[12] M. Visser Phys. Lett. **B159** (1985) 22.

[13] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84** (2000) 2778, hep-th/9911055.

[14] M. Sasaki, T. Shiromizu and K. Maeda, Phys. Rev. **D62** (2000) 024008hep-th/9912233.

[15] C. Csaki, J. Erlich, T. J. Hollowood and Y. Shirman, Nucl. Phys. **B581** (2000) 309, hep-th/0001033.

[16] S. B. Giddings, E. Katz and L. Randall, JHEP **03** (2000) 023, hep-th/0002091.

[17] I. Ya. Aref'eva, M. G. Ivanov, W. Mueck, K. S. Viswanathan and I. V. Volovich, Nucl.Phys. **B590** (2000) 273, hep-th/0004114.

[18] R. Maartens, Phys.Rev. **D62** (2000) 084023, hep-th/0004166 .

[19] D. Langlois, hep-th/0005025.

[20] C. van de Bruck, M. Dorca, R. Brandenberger and A. Lukas, hep-th/0005032

[21] D. Langlois, R. Maartens, D. Wands, Phys. Lett. **B489** (2000) 259, hep-th/0006007.

[22] Z. Kakushadze, hep-th/0008128.

[23] H. A. Bridgman, K. A. Malik and D. Wands, hep-th/0010133.

[24] N. Deruelle, T. Dolezel and J. Katz, hep-th/0010215.

[25] J. M. Bardeen, Phys Rev **D22** (1980) 1882.

[26] H. Kodama and M. Sasaki, Prog. Theor. Phys. Supp. **78** (1984) 1.

[27] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rep. **215** (1992) 203.

[28] H. A. Bridgman, K. A. Malik and D. Wands, to appear; C. van de Bruck and M. Dorca, *Cosmological Perturbations in Brane Worlds: Brane Bending and Anisotropic Stresses*, to appear.